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Optimization of Customized Pricing with Multiple Overlapping Competing Bids

Goutam Dutta¹
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Abstract

In this paper, we consider the case of project procurement where there is a single buyer and multiple sellers who are bidding. We consider one seller having one or more competitors. We formulate the pricing problem from the point of view of one seller having one or multiple competitors (say n). We also assume that based on past experience, we have some idea about the distribution of bid prices of the competitors. We consider uniform distribution to describe the bid price of the competitors. The prices of the competitors are pairwise mutually independent and the price range are either identical or different and overlapping. We consider maximizing the expected contribution. Assuming the contribution as a linear function of price we compute the conditions for maximization of the expected contribution to profit in case of n bidders. Further, we also compare the optimization results with simulation results.

Keywords: customized pricing, bid price, expected contribution, multiple competitors

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1. Introduction and Motivation

This paper is motivated by the experience of the first author in project procurement process. In procurement organization, in business to business marketing or conventional procurement of industrial products/equipment, there is a client organization which procures a product or equipment through competitive bidding. In response to these enquiries there are one or more sellers who are willing to bid for this product. We are looking at this problem from perspective of one seller (the first seller), say $S_0$, who is going to have either single or multiple competitors. We assume the history of the bidding price of all other competitors to be known and the distribution of bid prices are available. From the competitor’s interest, the bid prices can be spread across different ranges that can be described by a distribution. An earlier work (Phillips, 2011) looks into this problem of finding the optimal bid price when we have one seller and one or multiple competitors. In that work it is assumed that the bid prices follow a uniform distribution and the range of prices of all the competitors are identical. In this paper we extend the work when the bid prices of the competitors are not identical and the distribution of the prices follows two different distributions. We consider uniform and normal distribution of the bid prices.

We formulate the customized pricing decision as an optimization problem and then consider the method of solving this optimization problem to maximize the expected contribution to profit from each bid. The fundamental assumption is developing bid response function. In bid response function, we develop probability of win as a function of price. As the price goes up the probability of win goes down, however as the price goes up the contribution goes up also. We assume that the competing bids are statistically independent, and the competitors bid may be identical or overlapping. This paper makes fundamental contribution in three areas

1. While (Phillips, 2011) has discussed about competitive pricing for uniform distribution where the all the competitors’ bid price are identical (i.e. having the same upper and lower limits), we believe that this is the first attempt in developing a bid response function where we consider the expected contribution as a function of price when the competitors’ bid price having different price ranges are overlapping and the bid price of each have a distribution.
2. We also develop optimality conditions for expected contribution to profit in case of n competitors.

3. While Phillips (2011) has done the work for uniform distribution, we extend the work Normal distribution.

4. We compare our results with simulation.

This paper is organized as follows. The second section deals with the literature survey of competitive bid pricing. The third section describes customized pricing and the typical bidding process. The fourth section explains the theory of computing the optimal price. The fifth section consists of the generalized computation of optimal customized price, using any statistical distribution to describe the competitors bid prices, with subsections for single competitor model and multiple competitor models, in case of all competitors bidding identical prices and in case of competitors bidding different prices that may be overlapping. The sixth section consists of the optimization model that uses uniform distribution to describe the competitor’s bids, followed by subsections for single competitor model and multiple competitor models where all competitors bid price range are identical. The seventh section consists of the proposed model for multiple competitors using uniform distribution to describe the competitors bid prices with different price ranges leading to overlapping distribution. Further the seventh section contains two subsections describing the optimization model and optimality conditions for two competitors and four competitors respectively. The eighth section consists of the optimization model for the competitors bid prices following Normal distribution with subsections for single competitor model and multiple competitors’ model for both non-overlapping and overlapping distribution. Ninth section consists of computational results and the comparison of these with simulation results. The tenth section gives the relevance of this optimization model to management practice.

2. Literature Review

While looking at the literature of competitive bidding, we find that there are more than 500 publications in the area of auctions and bidding theory. There were three publication of literature survey in this area. A bibliography by Stark and Rothkopf (1979) discusses competitive bidding deals in construction contracts, oil exploration rights and securities. Secondly, to classify and describe various auction and bidding models, Engelbrecht (1980) created a framework based on the types of assumptions made for the models’ various parameters. Further, another study by Rothkopf and Harstad (1994) have analyzed bidding models that aid decision making in the bidding and auction strategy in real life transactions. This study also gives a brief summary of the direct use of these models, thus bringing closer the theory and direct application in decision making.
Several authors, Friedman (1956), Gates (1967) and Morin and Clough (1969) describe models in which price is the only criterion used to select a supplier. An initial study by Friedman (1956) explains a method that determines optimum bids in a competitive bidding situation by considering previous bidding patterns of the competitors with an estimated probability distribution of the cost. In this case, the numbers of competitors are either large or unknown. Hanssmann and Rivett (1959) explained a method of solution to the problem of making simultaneous bids for a number of objects with number of competitors competing to be unknown. Simmonds (1968) considered competitive bidding from the point of view of non-price features like quality, delivery, service or financing and their best combination that needs to be included in a bid. Dixie (1974) formulated a general predictive model for computing the probability of winning. Naert & Weverbergh (1978) examined the cost uncertainty in competitive bidding models and determined the parameters needed to estimate the value of information. King and Mercer (1985) determined the problems in bidding strategies and developed a methodology to estimate it. Further Benjamin and Meador (1979) presented a comparative study of Friedman and Gates bidding models. In order to determine the probability of a bidder winning a bid based upon the price, Bussey et. al. (1997) examined two novel pricing approaches. They modelled a multi-attribute utility theory to capture the bid selection behaviour of the client and possibility theory to capture the performance of competitors in the bid, thus determining the probability of the bidder winning the bid. A publication by Cagno et. al. (2001) describes a simulation approach based on the Analytic Hierarchy Process (AHP) to assess the probability of winning in a competitive bidding process where the bids are evaluated on a multiple criteria basis. For competitive bid selection Gindlesperger (2002) designed a system and method to obtain the lowest bid from information product vendors based on a database of vendor records.

Talus Solutions was the first company to develop a software system to optimize customized prices (Boyd, et. al., 2005). The first detailed treatment of an optimization approach to customized pricing for segmented markets was discussed by Phillips (2005). Agrawal and Ferguson (2007) apply a similar analytical approach to two examples of what they call customised-pricing bid-response models (CPBRMs) and compare the performance of segmented and unsegmented approaches. Phillips (2010) describes the application of optimal customized pricing in the specific context of consumer credit pricing.

Phillips (2005), Marn, et. al. (2007), Agrawal and Ferguson (2007) describe about customized pricing in a global telecommunications company, radio company and a medical testing company respectively and explain a way to maximize profitability.
The literature review for bidding can be distributed into the following four categories:

1) **Construction Management**: For construction contracting Gates (1967) developed competitive bidding strategies for optimizing profits considering no, one or many competitors who are either known or unknown. Gates also investigated some statistical distributions and its applications namely: Binomial Distribution, Gaussian Distribution and Poisson Distribution for forecasting the number of competitors, winning and precision of the bid. Further to assist the construction contractors in increasing their profits from competitive bidding, Morin and Clough (1969) developed a general computer program OPBID (optimum bid) that used the strategy of discrete probabilistic model for competitive bidding in construction industry. Benjamin (1969) reviewed the competitive bidding literature and then examined the competitive building problem in the building construction industry. He considered various elements in the building construction contracts and used various statistical tools to determine the optimal bid. A study by Whittaker (1970), describes an operational research study, applying decision theory and quantitative methods to competitive bidding problems in construction industry. Carr (1982), developed a general model of competitive bidding which is not limited by the assumptions on which Friedman’s and Gates’ models depend. Ioannou (1988) illustrated the correct use of symmetry in competitive bidding as a function of available information and also explained the differences among nonsymmetric states of information. Further the author presented the assumptions for general bidding model and validates Friedman’s model. A publication by Skitmore and Pemberton (1994) presents a multivariate approach to contract bidding mark-up strategies in construction industry. Another publication by Skitmore (2002) is concerned in predicting the probability of tendering the lowest bid in sealed bid auctions for construction contracts. The author uses binomial test to predict the lowest bidders and the average log score to predict the probability of each bidder being the lowest. Later Skitmore (2004) published another paper which was an extension of the above paper that also took into account the effects of outliers on predictions of the probability of winning sealed bid auctions in case of real construction contracts. Skitmore et. al. (2007) evaluated closed-bid competitive procurement auctions and determined the probability of placing a winning bid for a given mark-up level. They showed that Gates’ method will be valid if and only if bids can be described using statistical distribution that is related to Weibull distribution.

2) **Economics**: Milogram (1979), stated a convergence theorem for competitive bidding with a differential information that shows the winning bid converges in probability to the value of
the object at auction. The results of this convergence theorem depict the relationship between price and value. Later in 1981, Milogram developed a two stage market equilibrium model in which bidders act as price takers to get information before bidding and the equilibrium of that prices convey information. Higher equilibrium prices reveal information about the quality of the objects being sold than lower prices. Engelbrecht et. al. (1983) showed that at equilibrium the informed bidder’s distribution of bids is the same as the distribution of the maximum of the other bids. Also the equilibrium bid distributions and bidders’ expected profits are shown to vary in the parameters of the bidding game. Milogram (1985) also published a survey paper on the economics of competitive bidding.

3) Auction Theory and Game Theory: In order to value the subject of auctions Rothkopf (1969) developed a mathematical model for both highest-bid-wins and lowest-bid-wins auctions that deals with the uncertainty of the bidders. Again in 1980, Rothkopf considered symmetric bidding models where the bidders start with the same prior distribution on the value of the subject of the auction and studied the equilibrium linear bidding strategies when the bidder’s estimates are drawn from Weibull distribution.

4) Electricity markets and Power pricing for the competitive electricity generators: A publication by David and Wen (2000) is a literature survey based on the previous work done by more than 30 research publications. This work discusses strategic bidding in competitive electricity markets. Weber and Overbye (1999) presented a two-level optimization problem to analyse the bidding strategies in electricity markets. A new framework was built by Wen (2001) to optimize bidding strategies for power suppliers in an electricity market and model the imperfect information among competitive power generators.

While looking at the published literature we find that Phillips [2005, 2012] have discussed the single customer, multiple competitor’s model where probability distribution of the bid prices have been assumed to follow uniform distribution. The upper limit and the lower limit of the bid prices are identical i.e. the competitors bid price range are identical. However in the real world, all the competitors bid price range may not be identical. In this paper, we address this research gap by considering the bid price range of the competitors to be non-identical and extending the above concept using normal distribution to describe the competitors bid prices as shown in the Table 1 given below.
Table 1: Comparison of current research with Phillips (2011)

<table>
<thead>
<tr>
<th>Client</th>
<th>Distribution</th>
<th>Competitors</th>
<th>Upper and Lower limits</th>
<th>Optimality conditions</th>
<th>Simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phillips 2005, 2011</td>
<td>Uniform</td>
<td>1,2,…n</td>
<td>Identical</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Our Study</td>
<td>Uniform,</td>
<td>1,2,…n</td>
<td>Un-identical and overlapping</td>
<td>Yes, Piecewise</td>
<td>Yes</td>
</tr>
</tbody>
</table>

In this paper, we focus on formulating the customized pricing problem in case of multiple competitors or bidders using a uniform distribution with different upper and lower limits i.e. the competitors bid prices have an overlapping uniform distribution.

3. Problem of the seller in Competitive Pricing

Customized pricing refers to altering the price of goods or services based on customer preference factors where the seller can quote different prices to different customers. In a customized pricing situation a potential customer (buyer) approaches the sellers by inviting them to bid. The sellers respond to a bid by quoting a price for the bid, and the buyer chooses to determine the winning bidder. In customized pricing, the key element is the existence of bid or price quote. These type of purchasing decisions are common in project purchase where one has to buy an equipment. A typical example of such pricing will be a hospital trying to purchase MRI machines or a major construction company trying to purchase heavy duty cranes or an academic institution trying to purchase computers.

In this type of procurement situation, the buyer organization releases a request for proposal (RFP) from the sellers for solicit bids, with both technical and commercial details. In reply to this proposal, the quotes are offered by different sellers who are competing with each other. Of these quotes only a selected number of sellers are finally chosen for the final selection process based on the above technical and financial criterion. The buyer has to select the right seller, whereas the seller has to decide the optimal bid price. The project procurement bidding process consisting of one buyer and n bidding sellers is as shown in the figure 1.
4. Theory of Computing Optimal Price

We consider describing the prices of each competitor by a statistical distribution (uniform or normal). In the case of multiple competitors, we assume that the competing bids are statistically independent, and the competitors bid may be identical or overlapping.

4.1. Assumptions

a) The demand function is decreasing with price and there is an existence of inverse demand function.

b) Demand function is continuous and differentiable

c) The expected contribution function is finite and it is differentiable and may be piecewise non-linear.

d) It is possible to obtain a bid response function that calculates the probability of winning a bid for the buyer at different prices.

e) When the bid price of the one seller increases the probability of win decreases.

f) The bid price of all competing bids are pairwise and mutually independent

 g) The buyer selects only one seller

h) There is no tie among the sellers bid prices.

i) We look at the problem from the point of view of one initial seller who is bidding for a potential deal against one or more competitors

We consider a simple case where bidders bid on a potential deal by offering the price for a single product or service.
Notations:

\( S_0 \) = the initial seller
\( S_i \) = competitor sellers (i = 1,\ldots, n) bidding against seller \( S_0 \)
\( p \) = bid price or the price quoted by the seller \( S_0 \)
\( a_i \) = lower limit of price quoted by the seller \( S_i \)
\( b_i \) = upper limit of price quoted by the seller \( S_i \)
\( \rho(p) \) = the bid–response function or the probability that the seller \( S_0 \) wins the bid at price \( p \)
\( F_i(p) \) = cumulative distribution function of the \( i^{th} \) seller’s (\( S_i \)) bid price
\( f_i(p) \) = probability density function of the \( i^{th} \) seller’s (\( S_i \)) bid price
\( m(p) \) = deal contribution at price \( p \) of \( S_0 \)
\( \tau(p) \) = expected contribution at price \( p \) of \( S_0 \)
\( p^* \) = optimal price that maximizes expected contribution.
\( v \) = variable cost per unit

The seller \( S_0 \) wants to maximize his expected contribution from each bid. For a given price \( p \), expected contribution is expressed as

\[
\text{Expected contribution at price } p = \left( \text{Deal contribution at } p \right) \times \left( \text{Probability of winning the bid at } p \right)
\]

The first term on the RHS of the above equation 1 is the deal contribution i.e. the margin \( S_0 \) will realize if \( S_0 \) wins the bid. The second term on the RHS of the above equation 1 is the probability that \( S_0 \) wins the deal at a given price.

The general problem of customized pricing problem for a particular deal is

\[ \text{Maximize } \tau(p) = \rho(p)m(p) \]

The deal contribution is \( m(p) = (p - v) \) and hence, the total expected contribution is \( \tau(p) = (p-v)p(p) \) which is maximized. Optimize over \( p \) and find \( p^* \). The value of \( p^* \) maximizes \( \tau(p) \).

5. Generalized computation of the optimization problem

In general if there is a seller who is bidding for a potential deal against one or more bidding competitors, then the lower of all the bids wins in case of no tie among the bidders. Based on the past information about the competitor’s bidding behaviour, we assume the competitor’s bid price to follow certain distribution. In general case of two bidders, if \( p \) is the bid price quoted by the seller and \( q \) is the bid price quoted by his competitor then since the bids are independent, the seller wins the bid if \( p < q \) and the competitor wins the bid if \( p > q \). In the following subsections we consider the
optimization problem from the point of view of one seller $S_0$ with respect to single and multiple competitors $S_1, \ldots, S_n$.

5.1. Single competitor Model

In case of single competitor, suppose the competitor’s bid price follows a particular distribution with bid price within the limits $(a_1, b_1)$. If the seller $S_0$ bids a price below $a_1$ then the seller $S_0$ will surely win the bid as he knows that the competitor will bid more than $a_1$. Thus the probability of $S_0$ winning the bid, decreases between $a_1$ and $b_1$ and is zero if he bids above $b_1$. Thus the probability of win of $S_0$ is given as

$$
\rho(p) = \begin{cases} 
1, & \text{if } 0 \leq p \leq a_1 \\
1 - F(p), & \text{if } a_1 < p \leq b_1 \\
0, & \text{if } p > b_1
\end{cases}
$$

where, $1 - F(p) = \int_{p}^{b_1} f(x)dx$

The objective function is as follows:

Maximize $C = \tau(p) = (p - v)[1 - F(p)]$, if $a_1 < p \leq b_1$

**Theorem 1**: In case of single competitor $S_1$ bidding against $S_0$, the range of the bid price of $S_1$ is $(a_1, b_1)$. The expected contribution function $\tau(p)$ is continuous and differentiable in the interval $(a_1, b_1)$ and hence there exists an optimal value of price $p$ that maximizes the expected contribution function and the optimality conditions are given by

$$
p = \frac{1 - F(p)}{F'(p)} + v \quad \text{and} \quad p < \frac{-2F'(p)}{F''(p)} + v \quad \text{for } a_1 < p \leq b_1 \quad (2)
$$

5.2. Multiple competitor Model

There is usually more than one competitor to reduce our probability of winning a deal. If there are $n$ competitors, the seller wins the bid if he bids less than each one of the $n$ competitor’s bids.
5.2.1. Multiple competitors bidding identical price ranges

We consider n competitors bidding against seller $S_0$, such that all the n competitors bid prices are identical and follow a particular distribution. In other words, we assume identical beliefs about how the n competitors will bid. We assume the range of all the n competing bids to be $(a_1, b_1)$. The seller $S_0$ wins the bid if his bid price is less than each one of the n competitor’s bid prices. Thus the probability of win of $S_0$ is given as

$$
\rho(p) = \begin{cases} 
1, & \text{if } 0 \leq p \leq a_1 \\
(1 - F(p))^n, & \text{if } a_1 < p \leq b_1 \\
0, & \text{if } p > b_1
\end{cases}
$$

where, $1 - F(p) = \int_p^{b_1} f(x)dx$

The objective function is as follows:

Maximize $C = (p - v)(1 - F(p))^n, \text{if } a_1 < p \leq b_1$

**Theorem 2:** In case of n competitors $(S_1, ..., S_n)$ bidding against seller $S_0$, the range of all the n competitors $(S_1, ..., S_n)$ bid prices are identical in the interval $(a_1, b_1)$ i.e. the range of all the $F_i$ and $f_i$ ($i = 1, ..., n$) are independent and identically distributed (iid). The expected contribution function $\tau(p)$ is continuous and differentiable in the interval $(a_1, b_1)$ and hence there exists an optimal value of price $p$ that maximizes the expected contribution function and the optimality conditions are given by

$$
p = \frac{1 - F(p)}{nF'(p)} + v \text{ and } p < \frac{2F'(p)(1-F(p))}{(n-1)[F'(p)]^2 - F'(p)(1-F(p))} + v \text{ for } a_1 < p \leq b_1 \quad (3)
$$

5.2.2. Multiple competitors bidding different price ranges

**Case 1: Two competitors bidding with different prices ranges**

Suppose we consider seller $S_0$ bidding against two competitors $(S_1, S_2)$ such that their bids are independent and follow a particular distribution. Since the range of the two competitors bid prices are different, we observe an overlapping distribution.

The seller $S_0$ wins the bid if his bid price is less than each one of the two competitor’s bid prices. Thus probability of win of $S_0$ is given as
The objective function is as follows:

Maximize \( C = (p - v)(1 - F_1(p)) \), if \( a_1 < p \leq a_2 \)
\[ = (p - v)[1 - F_1(p)][1 - F_2(p)] \text{, if } a_2 < p \leq b_1 \]

**Theorem 3:** In case of two competitors, the range of the two competitors bid prices are \((a_1, b_1)\) and \((a_2, b_2)\) respectively that are overlapping. The expected contribution function \( \tau(p) \) is piecewise continuous and differentiable in the intervals \((a_1, a_2)\) and \((a_2, b_1)\) and hence there exists an optimal value of price \( p \) that maximizes the expected contribution function that lies either in the region \((a_1, a_2)\) or \((a_2, b_1)\) and the optimality conditions in each of these regions are given by

\[
\begin{align*}
 p &= \frac{1 - F_1(p)}{F_1(p)} + v \quad \text{and} \quad p < \frac{-2F_1(p)}{F_1(p)} + v \quad \text{for } a_1 < p \leq a_2 \quad \text{----------------- (4)} \\
 p &= \frac{[1 - F_1(p)][1 - F_2(p)]}{F_1(p)[1 - F_2(p)] + F_2(p)[1 - F_1(p)]} + v \quad \text{and} \\
 p &< \frac{2[1 - F_1(p)][1 - F_2(p)] + 2[1 - F_1(p)]F_1(p)}{2F_1(p)[1 - F_2(p)F_1(p)]} + v \quad \text{for } a_2 < p \leq b_1 \quad \text{----------------- (5)}
\end{align*}
\]

**Case 2: Four competitors bidding with different prices ranges**

Suppose we consider seller \( S_0 \) bidding against four competitors \( (S_1, \ldots, S_4) \) whose bid price ranges are different, independent and follows a particular distribution. Since the range of the four competitors bid prices are different, we observe an overlapping distribution.

The seller \( S_0 \) wins the bid if he bids less than each one of the four competitor’s bids. Thus probability of win of \( S_0 \) is given as

\[
\rho(p) = \begin{cases} 
1, & \text{if } 0 \leq p \leq a_1 \\
1 - F_1(p), & \text{if } a_1 < p \leq a_2 \\
[1 - F_1(p)][1 - F_2(p)], & \text{if } a_2 < p \leq a_3 \\
[1 - F_1(p)][1 - F_2(p)][1 - F_3(p)], & \text{if } a_3 < p \leq a_4 \\
[1 - F_1(p)][1 - F_2(p)][1 - F_3(p)][1 - F_4(p)], & \text{if } a_4 < p \leq b_1 \\
0, & \text{if } p > b_1
\end{cases}
\]

where, \( 1 - F_i(p) = \int_p^{b_i} f_i(x)dx \)
The objective function is as follows:

Maximize $C = (p - v)(1 - F_1(p))$, if $a_1 < p \leq a_2$

$$= (p - v)[1 - F_1(p)][1 - F_2(p)], \text{ if } a_2 < p \leq a_3$$

$$= (p - v)[1 - F_1(p)][1 - F_2(p)][1 - F_3(p)], \text{ if } a_3 < p \leq a_4$$

$$= (p - v)[1 - F_1(p)][1 - F_2(p)][1 - F_3(p)][1 - F_4(p)], \text{ if } a_4 < p \leq b_1$$

**Theorem 4:** In case of four competitors ($S_1, \ldots, S_4$) bidding against seller $S_0$, the range of the four competitors bid prices are $(a_1, b_1)$, $(a_2, b_2)$, $(a_3, b_3)$ and $(a_4, b_4)$ respectively, that are overlapping. The expected contribution function $\tau(p)$ is piecewise continuous and differentiable in the intervals $(a_1, a_2)$, $(a_2, a_3)$, $(a_3, a_4)$ and $(a_4, b_1)$ and hence there exists an optimal value of price $p$ that maximizes the expected contribution function that lies either in the region $(a_1, a_2)$, $(a_2, a_3)$, $(a_3, a_4)$ or $(a_4, b_1)$ and the optimality conditions in each of these regions are given by

$$p = \frac{1-F_4(p)}{F_4'(p)} + v \quad \text{and} \quad p < \frac{-2F_1'(p)}{F_1'(p)} + v \quad \text{for} \quad a_1 < p \leq a_2 \quad \text{(6)}$$

$$p = \frac{[1-F_4(p)][1-F_2(p)]}{F_4'(p)[1-F_2(p)]+F_2'(p)[1-F_4(p)]} + v \quad \text{and}$$

$$p < \frac{2[1-F_4(p)]F_2'(p)+2[1-F_2(p)]F_4'(p)}{2F_4'(p)F_2'(p)-[1-F_4(p)]F_2'(p)-[1-F_2(p)]F_4'(p)} + v \quad \text{for} \quad a_2 < p \leq a_3 \quad \text{(7)}$$

$$p = \frac{\prod_{i=1}^{4}[1-F_i(p)]}{\sum_{i\neq j,k}[1-F_i(p)][1-F_j(p)][1-F_k(p)]} + v \quad \text{and}$$

$$p < \frac{2 \prod_{i=1}^{4}[1-F_i(p)]}{\sum_{i=1}^{4}[1-F_i(p)][1-F_j(p)][1-F_k(p)] + 2 \prod_{i=1}^{4}[1-F_i(p)][1-F_j(p)][1-F_k(p)]} + v \quad \text{for} \quad a_3 < p \leq a_4 \quad \text{(8)}$$

$$p = \frac{\prod_{i=1}^{4}[1-F_i(p)]}{\sum_{i\neq j,k}[1-F_i(p)][1-F_j(p)][1-F_k(p)]} + v \quad \text{and}$$

$$p < \frac{2 \prod_{i=1}^{4}[1-F_i(p)]}{\sum_{i=1}^{4}[1-F_i(p)][1-F_j(p)][1-F_k(p)] + 2 \prod_{i=1}^{4}[1-F_i(p)][1-F_j(p)][1-F_k(p)]} + v \quad \text{for} \quad a_4 < p \leq b_1 \quad \text{(9)}$$
Case 3: n competitors bidding with different prices ranges

In general suppose we consider the seller $S_0$ bids against $n$ competitor’s bids that are different, independent and follow a particular distribution. Since the range of all the $n$ competitors bid prices are different, we observe an overlapping distribution.

The seller $S_0$ wins the bid if his bid price is less than each one of the $n$ competitor’s bid prices. Thus probability of win of seller $S_0$ is given as

$$\rho(p) = \begin{cases} 
1, & \text{if } 0 \leq p \leq a_1 \\
1 - F_1(p), & \text{if } a_1 < p \leq a_2 \\
[1 - F_1(p)][1 - F_2(p)], & \text{if } a_2 < p \leq a_3 \\
\vdots & \\
[1 - F_1(p)][1 - F_2(p)] \cdots [1 - F_{n-1}(p)], & \text{if } a_{n-1} < p \leq a_n \\
[1 - F_1(p)][1 - F_2(p)] \cdots [1 - F_n(p)], & \text{if } a_n < p \leq b_1 \\
0, & \text{if } p > b_1
\end{cases}$$

where, $1 - F_n(p) = \int_p^{b_n} f_n(x) \, dx$

The objective function is as follows:

Maximize $C = (p - v)(1 - F_1(p))$, if $a_1 < p \leq a_2$

$$= (p - v)[1 - F_1(p)][1 - F_2(p)], \text{ if } a_2 < p \leq a_3$$

$$= (p - v)[1 - F_1(p)][1 - F_2(p)] \cdots [1 - F_{n-1}(p)], \text{ if } a_{n-1} < p \leq a_n$$

$$= (p - v)[1 - F_1(p)][1 - F_2(p)] \cdots [1 - F_n(p)], \text{ if } a_n < p \leq b_1$$

Theorem 5: In case of $n$ competitors ($S_1, \ldots, S_n$) bidding against seller $S_0$, the range of the $n$ competitors bid prices are $(a_1, b_1)$, $(a_2, b_2)$, \ldots, $(a_n, b_n)$ respectively, that are overlapping. The expected contribution function $\tau(p)$ is piecewise continuous and differentiable in the intervals $(a_1, a_2)$, $(a_2, a_3)$, \ldots, $(a_{n-1}, a_n)$ or $(a_n, b_1)$ and hence there exists an optimal value of price $p$ that maximizes the expected contribution function that lies either in the region $(a_1, a_2)$, $(a_2, a_3)$, \ldots, $(a_{n-1}, a_n)$ or $(a_n, b_1)$ and the optimality conditions in each of these regions are given by

\[ p = \frac{1 - F_1(p)}{F_1'(p)} + v \quad \text{and} \quad p < \frac{-2F_1'(p)}{F_1''(p)} + v \quad \text{for } a_1 < p \leq a_2 \quad \text{(10)} \]

\[ p = \frac{[1 - F_1(p)][1 - F_2(p)]}{F_1'(p)[1 - F_2(p)] + F_2'(p)[1 - F_1(p)]} + v \quad \text{and} \]

\[ p < \frac{2[1 - F_1(p)]F_2'(p) + 2[1 - F_2(p)]F_1'(p)}{2F_1'(p)F_2'(p)[1 - F_1(p)][1 - F_2(p)]} + v \quad \text{for } a_2 < p \leq a_3 \quad \text{(11)} \]
6. Optimality conditions with Uniform Distribution

We now describe the competitor’s bid price by uniform distribution and study the optimization problem in case of single and multiple competitors.


In this case we consider seller $S_0$ bidding against one competitor seller $S_1$. Based on the past experience we describe the competitor’s ($S_1$) bid by a uniform distribution between $a_1$ and $b_1$ as reported in Phillips [2011]. We assume that $f(x)$ for the competitor is uniformly distributed between $(a_1, b_1)$ as shown in figure 2.
Then \( \rho(p) = 1 - F(p) \), where \( F(p) \) is the cumulative distribution function of the competing bid prices or it is the probability that the competing bid will be less than \( p \). If the seller \( S_0 \) bids below \( a_1 \) then the seller \( S_0 \) will surely win the bid as we know that the competitor will bid more than \( a_1 \). Thus as discussed earlier in section 5.1, the probability of \( S_0 \) winning the bid decreases linearly between \( a \) and \( b_1 \) and is zero if the seller bids above \( b_1 \).

Figure 3 shows the probability of \( S_0 \) winning the bid as a function of price when we assume that competitor’s bid follows the uniform distribution.

Thus, \( \rho(p) = \text{Equation of line AB, for all } p \), \( a_1 \leq p \leq b_1 \).

The general problem of customized pricing problem for a particular deal is

Maximize \( \tau(p) = \rho(p)m(p) \)
Then, \( m(p) = (p - v) \) and hence, the total deal contribution is \( \tau(p) = \rho(p)(p - v) \) which is maximized. Optimize over \( p \) and find \( p^* \). The value of \( p^* \) maximizes \( \tau(p) \).

### 6.2. Multiple Competitors: Uniform Distribution with identical price ranges

In a situation like these types of procurement process, there is usually a possibility of having more than one competitor in the bidding process. If there are \( n \) competitors \( (S_1, \ldots, S_n) \), seller \( S_0 \) will win the bid if the seller bids less than each one of the \( n \) competitor sellers \( S_1, \ldots, S_n \). We assume a uniform distribution to describe the competitors bid and the bids being independent. As explained by Phillips (2011), we assume that all the competitors have the same probability distribution function (pdf) of the bid price i.e. we assume identical distribution with same price ranges for all the \( n \) competitors \( (S_1, \ldots, S_n) \). The uniform distribution in this case is similar to as shown in figure 1. As described in section 5.2.1, the bid-response function, in this case, is

\[
\rho(p) = \begin{cases} 
1 & \text{if } p \leq a_1 \\
[1 - F(p)]^n & \text{if } a_1 < p < b_1 \\
0 & \text{if } p \geq b_1 
\end{cases}
\]

The probability of winning the bid of seller \( S_0 \) is along the curve AB if \( S_0 \) bids in the range \( (a_1, b_1) \).

![Figure 4. Probability of \( S_0 \) winning the bid as a function of price](attachment:image.png)

Thus, \( \rho(p) = \text{Equation of curve AB, for all } a_1 \leq p \leq b_1 \).

The general problem of customized pricing problem for a particular deal is

Maximize \( \tau(p) = \rho(p)(p - v) \)

We optimize over \( p \) and find optimal price \( p^* \) that maximizes \( \tau(p) \).
7. Optimality conditions for Multiple Competitors with Uniform Distribution

We consider the case of multiple competitors \((S_1, S_2, \ldots, S_n)\) bidding different price ranges and follows an uniform distribution. We assume the case where the pdf and cdf of the bid price of all the competitors are not identical. Since the competitors bid prices are different we observe an overlapping distribution.

7.1. Two Competitors: Uniform Distribution with Different and Overlapping Price Ranges

We consider the case of two competitors \((S_1, S_2)\) bidding against the seller \(S_0\). We consider competitor \(S_1\) bidding in the range \((a_1, b_1)\) and competitor \(S_2\) bidding in the range \((a_2, b_2)\) where \(a_1 < a_2 < b_1 < b_2\) and \(a_1 \neq a_2, b_1 \neq b_2\) as shown in figure 5. We assume the bid prices of both the competitors to be independent and uniformly distributed.

Suppose the seller \(S_0\) bids price \(p\), then \(S_0\) will win the bid if his bid is less than each of the two competing bids i.e. \(p < a_1\). Here the bids are overlapping, so the probability of win changes in each area and thus the formula describing the probability of win of \(S_0\) is different for the ranges between \(a_1\) and \(a_2\) and \(a_2\)-\(b_1\). If \(S_0\) bids below \$\(a_1\), then the seller \(S_0\) will surely win the bid as he knows that the competitors will bid more than \$\(a_1\). Thus the probability of winning the bid of \(S_0\) decreases linearly between \$\(a_1\) and \$\(a_2\), decreases along the curve between \$\(a_2\) and \$\(b_1\) and is zero if he bids above \$\(b_1\). Figure 6 shows \(S_0\)’s probability of winning the bid as a function of price along TABC, when we assume that competitor’s bid follows the uniform distribution with different price ranges.
We now consider the probability of win of $S_0$ corresponding to each region.

1) The probability of win $\rho(p)$ between $p = a_1$ and $p = a_2$ decreases along the straight line AB.
2) $\rho(p = a_1) = p_0 = 1$, $\rho(p = a_2) = p_1$ and $\rho(p = b_1) = 0$
3) $X \sim \text{unif}(a_1, b_1)$ and $Y \sim \text{unif}(a_2, b_2)$ with $a_1 < a_2 < b_1 < b_2$

$$P(Y > X, a_2 < Y < b_1) = \int_{a_2}^{b_1} \left( \int_{a_1}^{y} \frac{dx}{b_1 - a_1} \right) \frac{dy}{b_2 - a_2}$$

Equation of line AB: $\rho(p) = \frac{p_1(p - a_1) + p_0(a_2 - p)}{a_2 - a_1}$, $a_1 \leq p \leq a_2$

Equation of curve BC: $\rho(p) = \left( \frac{p_1(p_1 - p)}{b_1 - a_2} \right)^2$, $a_2 < p \leq b_1$

We Maximize $\tau(p) = \rho(p)m(p)$

where, $m(p) = (p - v)$ and hence, $\tau(p) = (p - v) \rho(p)$ is maximized. We optimize over $p$ and find $p^*$ that maximizes $\tau(p)$.

Let $C = \tau(p)$

The objective function is as follows:

$$C = \tau(p) = (p - v) \left( \frac{p_1(p - a_1) + p_0(a_2 - p)}{a_2 - a_1} \right), a_1 \leq p \leq a_2$$

$$= (p - v) \left( \frac{p_1(p_1 - p)}{b_1 - a_2} \right)^2, a_2 < p \leq b_1$$
Theorem 6: In case of two competitors \( (S_1, S_2) \) bidding against seller \( S_0 \), the bid price ranges of the competitors \( S_1 \) and \( S_2 \) are uniformly distributed in the intervals \( (a_1, b_1) \) and \( (a_2, b_2) \) respectively, such that the bid prices are not identical and overlapping. The expected contribution function \( \tau(p) \) is piecewise continuous and differentiable in the intervals \( (a_1, a_2) \) and \( (a_2, b_1) \) and hence there exists an optimal value of price \( p \) that maximizes the expected contribution function that lies either in the region \( (a_1, a_2) \) or \( (a_2, b_1) \) and the optimality conditions in each of these regions are given by

\[
p = \frac{p_1 a_1 - a_2 p_0}{2(p_1 - 1)} + \frac{v}{2} \quad \text{and} \quad p_1 < 1 \quad \text{for} \quad a_1 \leq p \leq a_2 \quad \text{------------------} (14)
\]

\[
p = b_1 \quad \text{or} \quad p = \frac{b_1}{3} + \frac{2v}{3} \quad \text{and} \quad p < \frac{2b_1}{3} + \frac{v}{3} \quad \text{for} \quad a_2 < p \leq b_1 \quad \text{------------------} (15)
\]

Theorem 7: In general when we consider the case of \( n \) competitors \( (S_1, \ldots, S_n) \) bidding against seller \( S_0 \), with their bid price ranges being independent and uniformly distributed in the intervals \( (a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n) \) respectively, such that the bid prices are not identical and overlapping. The expected contribution function \( \tau(p) \) is piecewise continuous and differentiable in the intervals \( (a_1, a_2), (a_2, a_3), \ldots, (a_n, b_1) \) and hence there exists an optimal value of price \( p \) that maximizes the expected contribution function that lies either in the region \( (a_1, a_2), (a_2, a_3), \ldots, (a_n, b_1) \) and the optimality conditions in each of these regions are given by
Table No. 2: First and Second Order Optimality conditions

<table>
<thead>
<tr>
<th>Region</th>
<th>First order conditions</th>
<th>Second order conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1 \leq p \leq a_2$</td>
<td>$p = \frac{-(p_0 a_2 - p_1 a_1)}{2(p_1 - 1)} + \frac{v}{2}$</td>
<td>$p_1 &lt; 1$</td>
</tr>
<tr>
<td>$a_2 &lt; p \leq a_3$</td>
<td>$p = \frac{-(p_1 a_2 - p_2 a_3)}{p_3 - p_1}$ or $p = \frac{-(p_1 a_3 - p_2 a_2)}{3(p_2 - p_1)} + \frac{2v}{3}$</td>
<td>$p_2 - p_1 &lt; 0$ or $p &lt; \frac{-2(p_1 a_2 - p_2 a_3)}{3(p_2 - p_1)} + \frac{2v}{6}$, $p_2 \neq p_1$</td>
</tr>
<tr>
<td>$a_3 &lt; p \leq a_4$</td>
<td>$p = \frac{-(p_2 a_4 - p_3 a_3)}{p_3 - p_2}$ or $p = \frac{-(p_2 a_3 - p_3 a_2)}{4(p_3 - p_2)} + \frac{3v}{4}$</td>
<td>$p &lt; \frac{-(p_2 a_4 - p_3 a_3)}{p_3 - p_2}$ or $p &lt; \frac{-(p_2 a_4 - p_3 a_3)}{2(p_3 - p_2)} + \frac{6v}{12}$, $p_3 \neq p_2$</td>
</tr>
<tr>
<td>$a_4 &lt; p \leq a_5$</td>
<td>$p = \frac{-(p_3 a_5 - p_4 a_4)}{p_4 - p_3}$ or $p = \frac{-(p_3 a_4 - p_4 a_3)}{5(p_4 - p_3)} + \frac{4v}{5}$</td>
<td>$p &lt; \frac{-(p_3 a_5 - p_4 a_4)}{p_4 - p_3}$ or $p &lt; \frac{-(p_3 a_5 - p_4 a_4)}{5(p_4 - p_3)} + \frac{12v}{20}$, $p_4 \neq p_3$</td>
</tr>
<tr>
<td>$a_{n-1} &lt; p \leq a_n$</td>
<td>$p = \frac{-(p_{n-2} a_n - p_{n-1} a_{n-1})}{p_{n-1} - p_{n-2}}$ or $p = \frac{-(p_{n-2} a_{n-1} - p_{n-1} a_{n-2})}{n(p_{n-1} - p_{n-2})} + \frac{(n-1)v}{n}$</td>
<td>$p &lt; \frac{-(p_{n-2} a_n - p_{n-1} a_{n-1})}{p_{n-1} - p_{n-2}}$ or $p &lt; \frac{-(p_{n-2} a_{n-1} - p_{n-1} a_{n-2})}{n(p_{n-1} - p_{n-2})} + \frac{(n-1)(n-2)v}{n(n-1)}$, $p_{n-1} \neq p_{n-2}$</td>
</tr>
<tr>
<td>$a_n &lt; p \leq b_1$</td>
<td>$p = \frac{b_1}{n+1} + \frac{nv}{(n+1)}$</td>
<td>$p &lt; b_1$ or $p &lt; \frac{2b_1}{n+1} + \frac{n(n-1)v}{n(n+1)}$</td>
</tr>
</tbody>
</table>

8. Computational Results

1) Two competitors (Uniform Distribution)

We consider an example of two competitors who bid for a product where the first competitor’s bid price is in the range $9000 - $14000 and second competitor’s bid price is in the range $12000 - $18000 wherein the competitors’ bids are not identical. In this case of two competitors, we will win the bid with probability 1 if our bid price is less than $9000 and win the bid with probability 0 if we bid more than $14000. Here we have $a_1 = 9000$, $b_1 = 14000$, $a_2 = 12000$, $b_2 = 18000$ where $a_1 < a_2 < b_1 < b_2$.

$$P(Y > X, 12000 < Y < 14000) = \int_{12000}^{14000} \frac{dx}{5000} = 0.4$$
Table No 3: Computed and Simulated results for n=2

<table>
<thead>
<tr>
<th>Value of v</th>
<th>Simulated value of price</th>
<th>Computed value of price from our model</th>
<th>Difference</th>
<th>Optimal Price Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>5000</td>
<td>9500</td>
<td>9500</td>
<td>0</td>
<td>9000 - 12000</td>
</tr>
<tr>
<td>6000</td>
<td>10000</td>
<td>10000</td>
<td>0</td>
<td>9000 - 12000</td>
</tr>
<tr>
<td>7000</td>
<td>10500</td>
<td>10500</td>
<td>0</td>
<td>9000 - 12000</td>
</tr>
<tr>
<td>8000</td>
<td>11000</td>
<td>11000</td>
<td>0</td>
<td>9000 - 12000</td>
</tr>
<tr>
<td>9000</td>
<td>11500</td>
<td>11500</td>
<td>0</td>
<td>9000 - 12000</td>
</tr>
<tr>
<td>10000</td>
<td>12000</td>
<td>12000</td>
<td>0</td>
<td>9000 - 12000</td>
</tr>
<tr>
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<td>12000</td>
<td>12500</td>
<td>500</td>
<td>12000 - 14000</td>
</tr>
<tr>
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<td>12667</td>
<td>13000</td>
<td>333</td>
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<tr>
<td>13000</td>
<td>13333</td>
<td>13500</td>
<td>167</td>
<td>12000 - 14000</td>
</tr>
</tbody>
</table>

2) Four competitors (Uniform Distribution)
We consider an example of four competitors who bid for a product where the first competitor’s bid price is in the range $5000 - $10000, second competitor’s bid price is in the range $7000 - $13000, third competitor’s bid price is in the range $8000 - $14000 and the fourth competitor’s bid price is in the range $9000 - $16000 wherein the competitors’ bids are not identical, hence they are overlapping.

Table No 4: Computed and Simulated results for n=4

<table>
<thead>
<tr>
<th>Value of v</th>
<th>Simulated value of price</th>
<th>Computed value of price from our model</th>
<th>Difference</th>
<th>Optimal Price Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>6000</td>
<td>6000</td>
<td>0</td>
<td>5000 - 7000</td>
</tr>
<tr>
<td>3000</td>
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<tr>
<td>6000</td>
<td>7000</td>
<td>7639</td>
<td>639</td>
<td>7000 - 8000</td>
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<td>7000</td>
<td>7999</td>
<td>8108</td>
<td>109</td>
<td>8000 - 9000</td>
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<td>8794</td>
<td>294</td>
<td>8000 - 9000</td>
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<tr>
<td>9000</td>
<td>9200</td>
<td>9421</td>
<td>221</td>
<td>9000 - 10000</td>
</tr>
</tbody>
</table>

9. Relevance to Management Practice

In our study of competitive bidding the buyer has all the information but seller may not get the information of the competitors. We assume that the purchasing behaviour of the buyer is rational and other things (like the technical specification being equal), buyer will buy the product which has the lowest bid. We also assume that all bidders being equal, the seller will try for a bid that increases his/her expected profit for the seller. It is possible that the seller will not be having all the information
about the other competitors but will know the range within which the competitors will bid. In such a situation, the practicing managers can get the optimal price to quote after making an assumption about the bidding behaviour of the competitors about the distribution of the bidding price of the competitors.

1) The seller will know the upper and lower limit of the bidding range of his/her competitors and does not know the distribution about the prices of competitors.

2) The seller also knows the distribution of the competitors bid price. The distribution of the bid price of the competitors could be uniform or normal.

In each case two decisions are important, first if the upper and lower limits are incorrect, will that affect the decision of buyer to switch from one seller to another seller. Second how much the probability of winning will change for the buyer. It is necessary to perform a sensitivity analysis in this case in both the distribution. The first condition is a must but if the upper and lower limits are provided incorrectly the problem is how incorrect is the bidding decision.

10. Conclusion and Extension for Further Research

1) Here we have considered the case where the competitor bids are independent but we can consider the case where the bids of the competitors are dependent. In this case we have to calculate the conditional probability.

2) The distributions of the bidding price have been assumed to be normal or uniform. The uniform distribution refers to linear demand response curve, normal distribution refers to logit demand curve. It is necessary to extend the work to other distribution like triangular and the beta distribution and find out how much is the difference in the optimal prices.

3) The work by Phillips [2011] describes how to construct a linear bid response curve for single competitor. This concept can be extended to develop a bid response curve for multiple competitors that will be useful in real world practice. In addition to that this will be tool that will help the practitioners to find the optimal price.
References


Appendix – 1

Statement 1:
Consider two values of price $p_1$ and $p_2$ such that $p_1 < p_2$
For $p_1 < p_2$, $p_1 - v < p_2 - v$, hence the contribution function is an increasing function. Thus the
contribution function is continuous, upward sloping function of $p$.
Again, for $p_1 < p_2$, $\rho(p_1) > \rho(p_2)$, hence the bid – response function or the probability of win $\rho(p)$
is a decreasing function. Thus $\rho(p)$ is continuous, downward sloping function of $p$.
The product of the bid-response and the contribution function is the expected contribution function
$\tau(p)$ which will be a smooth function that is a hill – shaped curve.
The expected contribution function being unconstrained, there exists an optimal price $p^*$ at the top of
the hill – shaped curve that maximizes the expected contribution.
The following shows the calculation of expected contribution as a function of price.

![Graph](image)

Figure A1: The Expected Contribution/Profit is the product of bid – response and contribution/
incremental profit. The expected contribution is calculated as a function of price and the
optimal price maximizing the expected contribution is shown as $p^*$

Source: Oxford Handbook of Pricing Management

Thus, for a decreasing bid – response curve and increasing contribution function, the expected
contribution function $C = \tau(p) = (p - v)\rho(p)$, is a smooth function of price having a unique optimal
price $p^*$ that maximizes the unconstrained expected contribution function.
Appendix – 2

Proof of Theorem 1: Will be provided on request
Proof of Theorem 2: Will be provided on request
Proof of Theorem 3: Will be provided on request
Proof of Theorem 4: Will be provided on request
Proof of Theorem 5: Will be provided on request
Proof of Theorem 6: Will be provided on request
Proof of Theorem 7: Will be provided on request